CHAPTER 8

UNSYMMETRICAL BENDING

8.1. Introduction

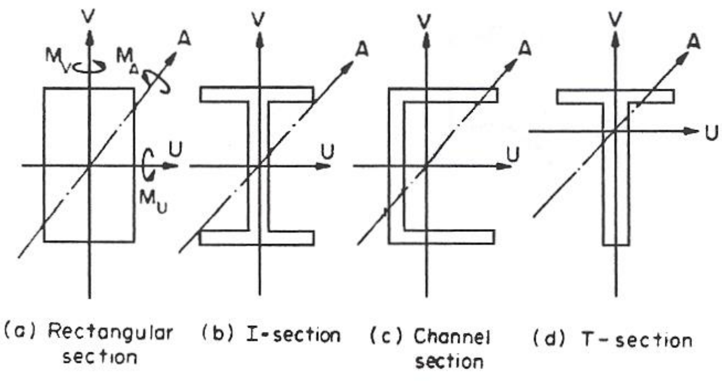
We know that the simple bending theory applies when bending takes place about an axis which is perpendicular to a plane of symmetry. If such an axis is drawn through the centroid of a section, and another mutually perpendicular to it also through the centroid, then these axes are principal axes. Thus a plane of symmetry is automatically a principal axis. Second moments of area of a cross-section about its principal axes are found to be maximum and minimum values, while the product second moment of area, $\int xy\,dA$, is found to be zero. All plane sections, whether they have an axis of symmetry or not, have two perpendicular axes about which the product second moment of area is zero. Principal axes are thus defined as the axes about which the product second moment of area is zero. Simple bending can then be taken as bending which takes place about a principal axis, moments being applied in a plane parallel to one such axis.

In general, however, moments are applied about a convenient axis in the cross-section; the plane containing the applied moment may not then be parallel to a principal axis. Such cases are termed “unsymmetrical” or “asymmetrical” bending. The most simple type of unsymmetrical bending problem is that of “skew” loading of sections containing at least one axis of symmetry, as in Fig. 1. This axis and the axis perpendicular to it are then principal axes and the term skew loading implies load applied at some angle to these principal axes. The method of solution in this case is to resolve the applied moment MA about some axis A into its components about the principal axes. Bending is then assumed to take place simultaneously about the two principal axes, the total stress being given by

$$\sigma = \frac{M_u v}{I_u} + \frac{M_v u}{I_v}$$

Fig.1

With unsymmetrical sections (e.g. angle-sections, Z-sections, etc.) the principal axes are not easily recognized and the second moments of area about the principal axes



are not easily found except by the use of special techniques to be introduced. In such cases an easier solution is obtained. Before proceeding with the various methods of solution of unsymmetrical bending problems, however, it is advisable to consider in some detail the concept of principal and product second moments of area.

8.2. <u>product second moment of area</u>

The product second moment of area of a section is defined as:

$$I_{xy} = \int xy\,dA$$

which reduces to I_{xy} = Ahk for a rectangle of area A and centroid distance h and k from the X and Y axes.

8.3. <u>Principal second moments of area</u>

The principal second moments of area are the maximum and minimum values for a section and they occur about the principal axes. **Product second moments of area about principal axes are zero.**

With a knowledge of l_{xx}, l_{yy} and l_{xy} for a given section, the principal values may be determined using either Mohr's or Land's circle construction.

The following relationships apply between the second moments of area about different axes:

$$I_u = \tfrac{1}{2}(I_{xx} + I_{yy}) + \tfrac{1}{2}(I_{xx} - I_{yy})\sec 2\theta$$
$$I_v = \tfrac{1}{2}(I_{xx} + I_{yy}) - \tfrac{1}{2}(I_{xx} - I_{yy})\sec 2\theta$$

where θ is the angle between the U and X axes, and is given by

$$\tan 2\theta = \frac{2I_{xy}}{(I_{yy} - I_{xx})}$$

Then $$I_u + I_v = I_{xx} + I_{yy}$$

8.4. <u>Second moment of area about the neutral axis</u>

The second moment of area about the neutral axis is given by

$$I_{\mathrm{N.A.}} = \tfrac{1}{2}(I_u + I_v) + \tfrac{1}{2}(I_u - I_v)\cos 2\alpha_u$$

α_u is the angle between the neutral axis (N.A.) and the U axis.

Also

$$\begin{aligned} I_{xx} &= I_u \cos^2\theta + I_v \sin^2\theta \\ I_{yy} &= I_v \cos^2\theta + I_u \sin^2\theta \\ I_{xy} &= \tfrac{1}{2}(I_v - I_u)\sin 2\theta \\ I_{xx} - I_{yy} &= (I_u - I_v)\cos 2\theta \end{aligned}$$

8.5. <u>skew loading and bending about principal axes</u>

For skew loading and other forms of bending about principal axes

$$\sigma = \frac{M_u v}{I_u} + \frac{M_v u}{I_v}$$

where M_u and M_v are the components of the applied moment about the U and V axes.
Alternatively, with $\sigma = Px + Qy$

$$\begin{aligned} M_{xx} &= PI_{xy} + QI_{xx} \\ M_{yy} &= -PI_{yy} - QI_{xy} \end{aligned}$$

Then the inclination of the N.A. to the X axis is given by

$$\tan\alpha = -\frac{P}{Q}$$

As a further alternative,

$$\sigma = \frac{M'n}{I_{N.A.}}$$

where M' is the component of the applied moment about the N.A., $I_{N.A.}$ is determined either from the momental ellipse or from the Mohr or Land constructions, and n is the perpendicular distance from the point in question to the N.A.

Deflections of unsymmetrical members are found by applying standard deflection formulaeto bending about either the principal axes or the N.A. taking care to use the correct component of load and the correct second moment of area value.

EXAMPLES

1.A rectangular-section beam 80 mm x 50 mm is arranged as a cantilever 1.3 m long and loaded at its free end with a load of 5 kN inclined at an angle of 30° to the vertical as shown in Fig. 2. Determine the position and magnitude of the greatest tensile stress in the section. What will be the vertical deflection at the end? E = 210 GN/m^2.

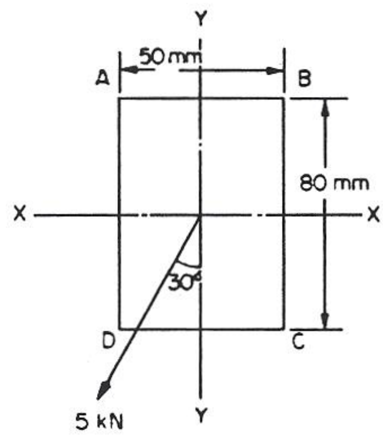


Fig.2

Solution

In the case of symmetrical sections such as this, subjected to skew loading, a solution is obtained by resolving the load into its components parallel to the two major axes and applying the bending theory simultaneously to both axes, i.e.

$$\sigma = \frac{M_{xx}y}{I_{xx}} \pm \frac{M_{yy}x}{I_{yy}}$$

Now the most highly stressed areas of the cantilever will be those at the built-in end where

$$M_{xx} = 5000\cos 30^\circ \times 1.3 = 5629 \text{ Nm}$$
$$M_{yy} = 5000\sin 30^\circ \times 1.3 = 3250 \text{ Nm}$$

The stresses on the short edges AB and DC resulting from bending about XX are then tensile on AB and compressive on DC.

$$\frac{M_{xx}}{I_{xx}}y = \frac{5629 \times 40 \times 10^{-3} \times 12}{50 \times 80^3 \times 10^{-12}} = 105.5\ \text{MN/m}^2$$

The stresses on the long edges AD and BC resulting from bending about YY are tensile on BC and compressive on AD.

$$\frac{M_{yy}}{I_{yy}}x = \frac{3250 \times 25 \times 10^{-3} \times 12}{80 \times 50^3 \times 10^{-12}} = 97.5\ \text{MN/m}^2$$

The maximum tensile stress will therefore occur at point B where the two tensile stresses add, i.e.

maximum tensile stress = 105.5 + 97.5 = 203 MN/m^2

The deflection at the free end of the cantilever is then given by

$$\delta = \frac{WL^3}{3EI}$$

Therefore deflection vertically (i.e. along the YY axis) is

$$\delta_v = \frac{(W\cos 30^\circ)L^3}{3EI_{xx}} = \frac{5000 \times 0.866 \times 1.3^3 \times 12}{3 \times 210 \times 10^9 \times 50 \times 80^3 \times 10^{-12}}$$
$$= 0.0071 = \mathbf{7.1\ mm}$$

2.A cantilever of length 1.2 m and of the cross section shown in Fig. 3 carries a vertical load of 10 kN at its outer end, the line of action being parallel with the longer leg and arranged to pass through the shear centre of the section (i.e. there is no twisting of the section). Working from first principles, find the stress set up in the section at points A, B and C, given that the centroid is located as shown. Determine also the angle of inclination of the N.A.

$$I_{xx} = 4 \times 10^{-6}\ \text{m}^4, \quad I_{yy} = 1.08 \times 10^{-6}\ \text{m}^4$$

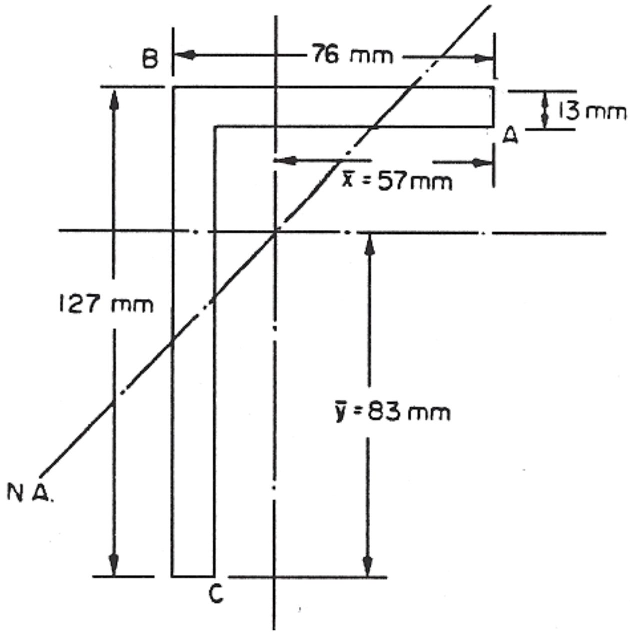


Fig. 3

Solution

The product second moment of area of the section is given by

$$\begin{aligned} I_{xy} &= \Sigma Ahk \\ &= \{76 \times 13(\tfrac{1}{2} \times 76 - 19)(44 - \tfrac{1}{2} \times 13) \\ &\quad + 114 \times 13[-(83 - \tfrac{1}{2} \times 114)][-(19 - \tfrac{1}{2} \times 13)]\}10^{-12} \end{aligned}$$

$$= (0.704 + 0.482)10^{-6} = 1.186 \times 10^{-6} \text{ m}^4$$

We know that
$$M_x = PI_{xy} + QI_{xx} = 10\,000 \times 1.2 = 12\,000$$

i.e.
$$1.186P + 4Q = 12\,000 \times 10^6$$

Since the load is vertical there will be no moment about the Y axis and eqn. below gives

$$M_y = -PI_{yy} - QI_{xy} = 0$$

$$\therefore \quad -1.08P - 1.186Q = 0$$

$$\therefore \quad \frac{P}{Q} = -\frac{1.186}{1.08} = -1.098$$

But the angle of inclination of the N.A. is given by eqn.

$$\tan \alpha_{\mathrm{N.A.}} = -\frac{P}{Q} = 1.098$$

i.e. $\alpha_{\mathrm{N.A.}} = \mathbf{47°41'}$

Substituting $P = -1.098Q$ in eqn. (1),

$$1.186(-1.098Q) + 4Q = 12\,000 \times 10^6$$

$$\therefore \quad Q = \frac{12\,000 \times 10^6}{2.69} = 4460 \times 10^6$$

$$\therefore \quad P = -4897 \times 10^6$$

If the N.A. is drawn as shown in Fig. 3 at an angle of 47°41' to the XX axis through (the centroid of the section, then this is the axis about which bending takes place. The points of maximum stress are then obtained by inspection as the points which are the maximum perpendicular distance from the N.A.

Thus B is the point of maximum tensile stress and C the point of maximum compressive stress.

We know that the stress at any point is given by $\sigma = Px + Qy$

$\therefore$ stress at A = - 4897 x 10^6(57 x 10^{-3}) + 4460 x 10^6(31 x $l0^{-3}$)

= -141 MN/m^2 (compressive)

stress at B = - 4897 x 10^6(-19 x 10^{-3}) + 4460 x 106(44 x 10)

= 289 MN/m^2 (tensile)

stress at C = - 4897 x 10^6(- 6 x 10^{-3}) + 4460 x 10^6(- 83 x 10^{-3})

= - 341 MN/m^2 (compressive)

3.(a) A horizontal cantilever 2 m long is constructed from the Z-section shown in Fig. 4. A load of 10 kN is applied to the end of the cantilever

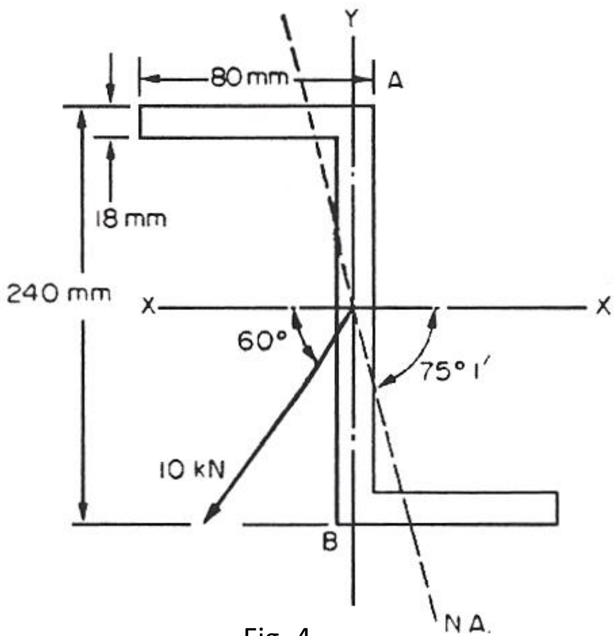


Fig. 4

at an angle of 60° to the horizontal asshown. Assuming that no twisting moment is applied to the section, determine the stresses at points A and B. (I_{xx} = 48.3 x 10^{-6} m^4, I_{yy} = 4.4 x 10^{-6} m^4.)

(b) Determine the principal second moments of area of the section and hence, by applying the simple bending theory about each principal axis, check the answers obtained in part (a).

(c) What will be the deflection of the end of the cantilever? E = 200 GN/m^2.

Solution

(a) For this section I_{xy} for the web is zero since its centroid lies on both axes and hence h and k are both zero. The contributions to I_{xy} of the other two portions will be negative since in

$$\therefore \quad I_{xy} = -2(80 \times 18)(40-9)(120-9)10^{-12}$$
$$= -9.91 \times 10^{-6}\ \text{m}^4$$

Now, at the built-in end,

$$M_x = +10\,000 \sin 60^\circ \times 2 = +17\,320\ \text{Nm}$$
$$M_y = -10\,000 \cos 60^\circ \times 2 = -10\,000\ \text{Nm}$$

Substituting in eqns. (1.20) and (1.21),

$$17\,320 = PI_{xy} + QI_{xx} = (-9.91P + 48.3Q)10^{-6}$$
$$-10\,000 = -PI_{yy} - QI_{xy} = (-4.4P + 9.91Q)10^{-6}$$
$$\therefore \quad 1.732 \times 10^{10} = -9.91P + 48.3Q \quad (1)$$
$$-1 \times 10^{10} = -4.4P + 9.91Q \quad (2)$$

$(1) \times \frac{4.4}{9.91}$,

$$0.769 \times 10^{10} = -4.4P + 21.45Q \quad (3)$$

both cases either h or k is negative.

(3) − (2),

$$1.769 \times 10^{10} = 11.54Q$$
$$\therefore \quad Q = 1533 \times 10^6$$

and substituting in (2) gives

$$P = 5725 \times 10^6$$

The inclination of the N.A. relative to the X axis is then given by

$$\tan \alpha_{\text{N.A.}} = -\frac{P}{Q} = -\frac{5725}{1533} = -3.735$$
$$\alpha_{\text{N.A.}} = \mathbf{-75^\circ 1'}$$

This has been added to Fig. 4 and indicates that the points A and B are on either side of the N.A. and equidistant from it. Stresses at A and B are therefore of equal magnitude but opposite

sign.

$$\sigma = Px + Qy$$

Now $\therefore$ stress at A $= 5725 \times 10^6 \times 9 \times 10^{-3} + 1533 \times 10^6 \times 120 \times 10^{-3}$

$$= \mathbf{235 \ \ MN/m^2 \ (tensile)}$$

Similarly,

$$\text{stress at } B = \mathbf{235 \ MN/m^2 \ (compressive)}$$

(b) The principal second moments of area may be found from Mohr's circle as shown in Fig. 5.

i.e.
$$I_u, I_v = \tfrac{1}{2}(I_{xx} + I_{yy}) \pm \tfrac{1}{2}(I_{xx} - I_{yy}) \sec 2\theta$$

with
$$\tan 2\theta = \frac{2I_{xy}}{I_{yy} - I_{xx}} = \frac{-2 \times 9.91 \times 10^{-6}}{(4.4 - 48.3)10^{-6}}$$

$$= 0.451$$

$$\therefore \quad 2\theta = 24^\circ 18', \ \ \theta = 12^\circ 9'$$

$$\therefore \quad I_u, I_v = \tfrac{1}{2}[(48.3 + 4.4) \pm (48.3 - 4.4)1.0972]10^{-6}$$

$$= \tfrac{1}{2}[52.7 \pm 48.17]10^{-6}$$

$$\therefore \quad I_u = 50.43 \times 10^{-6} \ \mathrm{m}^4$$

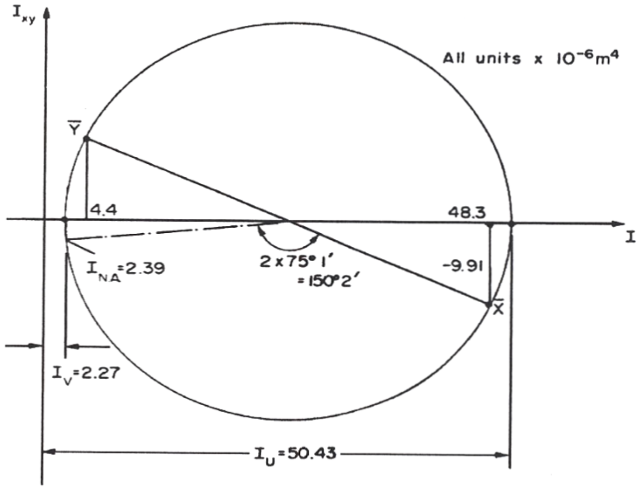
$$I_v = 2.27 \times 10^{-6} \ \mathrm{m}^4$$

The required stresses can now be obtained from eqn. below.

$$\sigma = \frac{M_v u}{I_v} + \frac{M_u v}{I_u}$$

Now
$$M_u = 10\,000 \sin(60^\circ - 12^\circ 9') \times 2$$

$$= 10\,000 \sin 47^\circ 51' \times 2 = 14\,828 \ \mathrm{Nm}$$



and $$M_v = 10\,000 \cos 47°51' \times 2 = 13\,422 \text{ Nm}$$

and, for A,

$$u = x\cos\theta + y\sin\theta = (9 \times 0.9776) + (120 \times 0.2105)$$
$$= 34.05 \text{ mm}$$
$$v = y\cos\theta - x\sin\theta = (120 \times 0.9776) - (9 \times 0.2105)$$
$$= 115.4 \text{ mm}$$

$$\therefore \quad \sigma = \frac{14\,828 \times 115.4 \times 10^{-3}}{50.43 \times 10^{-6}} + \frac{13\,422 \times 34.05 \times 10^{-3}}{2.27 \times 10^{-6}}$$
$$= \mathbf{235\ MN/m^2} \text{ as before.}$$

(c) The deflection at the free end of a cantilever is given by

$$\delta = \frac{WL^3}{3EI}$$

Therefore component of deflection perpendicular to the V axis

$$\delta_v = \frac{W_v L^3}{3EI_v} = \frac{10\,000 \cos 47°51' \times 2^3}{3 \times 200 \times 10^9 \times 2.27 \times 10^{-6}}$$
$$= 39.4 \times 10^{-3} = 39.4 \text{ mm}$$

and component of deflection perpendicular to the U axis

$$\delta_u = \frac{W_u L^3}{3EI_u} = \frac{10\,000 \sin 47°51' \times 2^3}{3 \times 200 \times 10^9 \times 50.43 \times 10^{-6}}$$

$$= 1.96 \times 10^{-3} = 1.96 \text{ mm}$$

The total deflection is then given by

$$= \sqrt{(\delta_u^2 + \delta_v^2)} = 10^{-3}\sqrt{(39.4^2 + 1.96^2)} = 39.45 \times 10^{-3}$$

$$= \mathbf{39.45\ mm}$$

Alternatively, since bending actually occurs about the N.A., the deflection can be found from

$$\delta = \frac{W_{\text{N.A.}} L^3}{3EI_{\text{N.A.}}}$$

its direction being normal to the N.A.

From Mohr's circle of Fig. 5, $I_{\text{N.A.}} = 2.39 \times 10^{-6}$ m^4

$$\therefore \qquad \delta = \frac{10\,000 \sin(30° + 14°59') \times 2^3}{3 \times 200 \times 10^9 \times 2.39 \times 10^{-6}} = 39.44 \times 10^{-3}$$

$$= \mathbf{39.44\ mm}$$